On Numerical Integration of Equally Spaced Hybrid Method for Third-Order Initial Value Problems Using Orthogonal Trial Function

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ABSTRACT

The study addresses the essential role of Differential Equations (DES) in modelling physical phenomena and acknowledges the challenge posed by the inability to solve many des analytically. To overcome this, efficient numerical and approximation methods are necessary. The focus is on constructing a family of orthogonal polynomials valid in the interval [-1, 1] with a specific weight function. The hybrid two-step equally spaced method (htepm), employs collocation and interpolation techniques. On investigation of the fundamental properties of the method, findings reveal that the proposed schemes are consistent, zero-stable, and consequently convergent. Upon implementation, the study establishes the numerical superiority of the HTEPM over existing methods through rigorous numerical evaluations and comparisons. This suggests that the proposed method offers improved performance in solving DES within the specified context.

Keywords: Collocation; Interpolation; Orthogonal Polynomials, Block Method, DEs;

1.0 INTRODUCTION

integration techniques play a pivotal role in solving differential equations, especially in scenarios where anaNumericallytical solutions are unattainable. This study focuses on the numerical integration of third-order initial value problems (IVPs) utilizing an equally spaced hybrid method (HTEPM), augmented with orthogonal trial functions via collocation and interpolation technique.

Our main goal is to derive a new class of polynomials that may be used to a wide range of situations. Several writers have proposed ways for handling initial value problems because they want to improve the accuracy and efficiency of numerical approaches ([24] ,[26],[30]). Our goal is to create a class of orthogonal polynomials in this work that will be used as trial functions to construct numerical methods for a class of initial value issues that look like this:

$$y^{m}(x) = f(x, y, y', \dots y^{m-1}) \qquad (1)$$

$$y^{r}(x_0) = y_r \,, r = 0, 1, \dots, k-1$$

Specifically, we considered the case $m = 3$,

The analytical solution of many of such problems does not exist. Thus, the need for formulation of numerical scheme to integrate (1) becomes neccesary.

Recently, there has been a focus on exploring the numerical solution of Ordinary Differential Equations (ODEs) (1) for cases where m equals 1, 2, and 3 using collocation methods, as evidenced by studies referenced in [1], [3], [4], [5], and [18]]]].

More recently, [8], [9], [14], and [19] have developed various numerical methods and explored diverse trial functions, contributing to the ongoing advancement of the field

Authors in both references [14] and [21] embraced the self-starting approach, employing Chebyshev Polynomials to formulate a series of algorithms. The numerical solutions derived through their methodologies are noteworthy, as they frequently converge to exact solutions at numerous instances. In what follows, we shall construct a set of orthogonal polynomials valid in interval [0, 1] with respect to weight function $w(x) = 1 + \frac{x}{2}$ which are serve as trial functions to derive a block method that provides direct solution to (1).

2.0 CONSTRUCTION OF ORTHOGONAL BASIS FUNCTIONS

Let the function $q_n(x)$ defined as

$$q_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (2)$$

where $C_r^{(n)}$'s are the orthogonal coefficients and $q_n(x)$ satisfies the inner product

$$< q_m(x), q_n(x) > = \int_a^b w(x) q_m(x) q_n(x) dx = 0, \quad m \neq n, [-1,1] \qquad (3)$$

For the purpose of constructing the basis function, we use additional property that

$$q_n(1) = 1 \qquad (4)$$

For $n = 0$ in (2),

$$q_0(x) = C_0^{(0)}$$

From (4),

$$q_0(1) = C_0^{(0)} = 1$$

Hence,

$$q_0(x) = 1$$

For $n = 1$ in (2),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \qquad (5)$$

By definition (4), (5) gives

$$C_0^{(1)} + C_1^{(1)} = 1 \qquad (6)$$

and

$$< q_0, q_1 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_1(x) dx \qquad (7)$$

which implies

$$\frac{5}{4} C_0^{(1)} + \frac{2}{3} C_1^{(1)} = 0 \qquad (8)$$

Solving (6) and (10) and substituting the outcomes into (5), we have

$$q_1(x) = \frac{1}{7}(15x - 8) \qquad (9)$$

When $n = 2$ in (2),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \qquad (10)$$

By definition (4), (10) gives

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \qquad (11)$$

and

$$< q_0, q_2 > = \int_0^1 \left(1+\frac{x}{2}\right) q_0(x)q_2(x)dx = 0 \quad (12)$$

which implies

$$\frac{5}{4}C_0^{(2)} + \frac{2}{3}C_1^{(2)} + \frac{11}{24}C_2^{(2)} = 0 \quad (13)$$

Also

$$< q_1, q_2 > = \int_0^1 \left(1+\frac{x}{2}\right) q_1(x)q_2(x)dx \quad (14)$$

which gives

$$\frac{37}{168}C_1^{(2)} + \frac{19}{84}C_2^{(2)} = 0 \quad (15)$$

Solving (11), (13), (15) and substituting the resulting values into (10), we have

$$q_2(x) = \frac{1}{57}(370x^2 - 380x + 67) \quad (16)$$

When $n = 3$ in (2),

$$q_3(x) = C_0^{(3)} + C_1^{(3)}x + C_2^{(3)}x^2 + C_3^{(3)}x^3 \quad (17)$$

By definition (4), (17) gives

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \qquad (18)$$

$$< q_0, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_3(x) dx = 0 \quad (19)$$

which implies

$$\frac{5}{4} C_0^{(3)} + \frac{2}{3} C_1^{(3)} + \frac{11}{24} C_2^{(3)} + \frac{7}{20} C_3^{(3)} = 0 \qquad (20)$$

$$< q_1, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_1(x) q_3(x) dx = 0 \quad (21)$$

This leads to

$$\frac{37}{168} C_1^{(3)} + \frac{19}{84} C_2^{(3)} + \frac{29}{140} C_3^{(3)} = 0 \qquad (22)$$

$$< q_2, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_2(x) q_3(x) dx = 0 \quad (23)$$

Solving (18),(20) and (22) and substituting the resulting values into (17), we obtain

$$q_3(x) = \frac{1}{491}(10675x^3 - 16290x^2 + 6690x - 584) \qquad (24)$$

In the same vein, $q_n(x), n \geq 4$ are developed. The next three polynomials which are used in this work are listed hereunder.

$$q_4(x) = \frac{1}{4361}(332766x^4 - 674072x^3 + 440874x^2 - 100428x + 5221)$$

$$q_5(x) = \frac{1}{7899}(2173710x^5 - 5489736x^4 + 4942812x^3 - 1884904x^2 + 275513x - 9496)$$

$$q_6(x) = \frac{1}{72509}(73254324x^6 - 221626152x^5 + 254436138x^4 - 137426374x^3 + 34913052x^2 - 3565896x + 87419)$$

2.1 Formulation of the Numerical Integration

We shall seek an approximation of the form

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \qquad (25)$$

where $q_r(x)$ is the orthogonal polynomials derived.

Our objective is to derive a two-step continuous hybrid linear multistep method in the sub-interval $[x_n, x_{n+p}]$ of [a, b] where $x = \frac{2X-2x_n-ph}{ph}$ and p varies as the method to be derived. For this case

$p = 2.$

The procedure involves interpolating (25) at $x = x_{k+i}$, $i = 0, \frac{1}{2}, 1$and collocating the third derivative of (25) at $x = x_{k+i}$, $i = 0, \frac{1}{2}, 1, \frac{3}{2}$ *and* 2.

The $a_r (0 \leq r \leq 7)$, from the resulting system of equations are obtained and substituted into (25) to have the continuous equation

Evaluating equation (26) at $x = x_{k+\frac{3}{2}}$ and $x = x_{k+2}$ yield the following main methods as

$$y_{k+\frac{3}{2}} = y_k - 3y_{k+\frac{1}{2}} + 3y_{k+1} + \frac{h^3}{1920}\left(f_k + 116f_{k+\frac{1}{2}} + 126f_{k+1} - 4f_{k+\frac{3}{2}} + f_{k+2}\right) \quad (27)$$

$$y_{k+2} = 3y_k - 8y_{k+\frac{1}{2}} + 6y_{k+1} + \frac{h^3}{480}\left(f_k + 86f_{k+\frac{1}{2}} + 126f_{k+1} + 26f_{k+\frac{3}{2}} + f_{k+2}\right) \quad (28)$$

The general block formular proposed in[10] in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\tau}df(y_m) + h^{\mu-\tau}bF(y_m) \quad (29)$$

shall be adopted inorder to develop the block method from the continuous scheme.

Evaluating the first and second derivatives of (26) at $x = x_{k+i},\ i = 0, \frac{1}{2}, 1, \frac{3}{2}\ and\ 2$ and substituting the resulting equations and the main methods (27), (28) into (29) and solving simultaneously gives a block formula represented as

$$y_{k+\frac{1}{2}} = y_k + \frac{1}{2}hy'_k + \frac{1}{8}h^2y''_k + \frac{113}{8960}h^3f_k - \frac{103}{13440}h^3f_{k+1} - \frac{47}{80640}h^3f_{k+2} + \frac{107}{8064}h^3f_{k+\frac{1}{2}} + \frac{43}{13440}h^3f_{k+\frac{3}{2}}$$

$$y_{k+1} = y_k + hy'_k + \frac{1}{2}h^2y''_k + \frac{331}{5040}h^3f_k - \frac{1}{21}h^3f_{k+1} - \frac{19}{5040}h^3f_{k+2} + \frac{83}{630}h^3f_{k+\frac{1}{2}} + \frac{13}{630}h^3f_{k+\frac{3}{2}}$$

$$y_{k+\frac{3}{2}} = y_k + \frac{3}{2}hy'_k + \frac{9}{8}h^2y''_k + \frac{1431}{8960}h^3f_k - \frac{243}{4480}h^3f_{k+1} - \frac{81}{8960}h^3f_{k+2} + \frac{1863}{4480}h^3f_{k+\frac{1}{2}} + \frac{45}{896}h^3f_{k+\frac{3}{2}}$$

$$y_{k+2} = y_k + 2hy'_k + 2h^2y''_k + \frac{31}{105}h^3f_k + \frac{4}{105}h^3f_{k+1} - \frac{1}{63}h^3f_{k+2} + \frac{272}{315}h^3f_{k+\frac{1}{2}} + \frac{16}{105}h^3f_{k+\frac{3}{2}}$$

$$y'_{k+\frac{1}{2}} = hy'_k + h^2y''_k + \frac{53}{360}h^3f_k - \frac{1}{12}h^3f_{k+1} - \frac{1}{120}h^3f_{k+2} + \frac{2}{5}h^3f_{k+\frac{1}{2}} + \frac{2}{45}h^3f_{k+\frac{3}{2}}$$

$$y'_{k+1} = hy'_k + \frac{3}{2}h^2y''_k + \frac{147}{640}h^3f_k + \frac{27}{320}h^3f_{k+1} - \frac{9}{640}h^3f_{k+2} + \frac{117}{160}h^3f_{k+\frac{1}{2}} + \frac{3}{32}h^3f_{k+\frac{3}{2}}$$

$$y'_{k+\frac{3}{2}} = hy'_k + \frac{1}{2}h^2y''_k + \frac{367}{5760}h^3f_k - \frac{47}{960}h^3f_{k+1} - \frac{7}{1960}h^3f_{k+2} + \frac{3}{32}h^3f_{k+\frac{1}{2}} + \frac{29}{1440}h^3f_{k+\frac{3}{2}}$$

$$y'_{k+2} = hy'_k + 2h^2y''_k + \frac{14}{45}h^3f_k + \frac{4}{15}h^3f_{k+1} + \frac{16}{15}h^3f_{k+\frac{1}{2}} + \frac{16}{45}h^3f_{k+\frac{3}{2}}$$

$$y''_{k+\frac{1}{2}} = h^2 y''_k + \frac{29}{180} h^3 f_k + \frac{2}{15} h^3 f_{k+1} - \frac{1}{180} h^3 f_{k+2} + \frac{31}{45} h^3 f_{k+\frac{1}{2}} + \frac{1}{45} h^3 f_{k+\frac{3}{2}}$$

$$y''_{k+1} = h^2 y''_k + \frac{27}{160} h^3 f_k + \frac{9}{20} h^3 f_{k+1} - \frac{3}{160} h^3 f_{k+2} + \frac{51}{80} h^3 f_{k+\frac{1}{2}} + \frac{21}{80} h^3 f_{k+\frac{3}{2}}$$

$$y''_{k+\frac{3}{2}} = h^2 y''_k + \frac{251}{1440} h^3 f_k - \frac{11}{60} h^3 f_{k+1} - \frac{19}{1440} h^3 f_{k+2} + \frac{323}{720} h^3 f_{k+\frac{1}{2}} + \frac{53}{720} h^3 f_{k+\frac{3}{2}}$$

$$y''_{k+2} = h^2 y''_k + \frac{7}{45} h^3 f_k + \frac{4}{15} h^3 f_{k+1} + \frac{7}{45} h^3 f_{k+2} + \frac{32}{45} h^3 f_{k+\frac{1}{2}} + \frac{32}{45} h^3 f_{k+\frac{3}{2}} \qquad (30)$$

4.0 Numerical Applications

We consider here the application of the derived schemes to three test problems for the efficiency and accuracy of the method implemented as block method

Problem 4.1.1: (A non-linear problem)

$$y^2 y''' = 1, \quad y(0) = 1 \quad y'(0) = 1, \quad y''(0) = 1, \quad h = 0.1$$

Source: [11]

The above problem was derived by Tanner to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface.

Problem 4.1.2 Non-linear Blasius Equation (Application Problem)

$$2y''' + yy'' = 0$$

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$$

The exact solution does not exist.

Source: [2]

Problem 4.1.3 Non-linear Genesio Equation (Application Problem)

Here we consider the nonlinear chaotic system from Genesio and Tesi (1992)

$$x''' + Ax'' + Bx' - f(x(t)) = 0$$

with

$$f(x(t)) = -Cx(t) + x^2(t)$$

that is subject to the following initial conditions:

$$x(0) = 0.2, \quad x'(0) = -0.3, \quad x''(0) = 0.1, \quad t \in [0, b],$$

where $A = 1.2, \quad B = 2.29 \quad and \quad C = 6$ are positive constants that satisfied $AB < C$ for the existence of the solution.

Source: [15]

TABLE 4.1.1: COMPARING THE SOLUTION OF THE APPROXIMATE AND THE EXISTING METHOD FOR PROBLEM 4.1.1

X	Exact Solution	Result of New Method	Error in New Method Order $P = 5$	Error in Tanner (1979) Order $P = 4$
0.2	1.22121001337746352620	1.22121000401169152860	$9.36577199 \times 10^{-09}$	$2.40500000 \times 10^{-05}$
0.4	1.48883473296637175650	1.48883477650976717200	$4.35433954 \times 10^{-08}$	$7.71670000 \times 10^{-05}$
0.6	1.80736134919720764840	1.80736138815762258600	$3.89604149 \times 10^{-08}$	$7.94945000 \times 10^{-06}$
0.8	2.17981922624938085950	2.17981921459690672930	$1.16524741 \times 10^{-08}$	$4.34949000 \times 10^{-03}$
1.0	2.60827491835217941000	2.60827483474311471480	$8.36090470 \times 10^{-08}$	$1.83199620 \times 10^{-02}$

TABLE 4.1.2: COMPARING THE SOLUTION OF THE APPROXIMATE AND THE EXISTING METHOD FOR PROBLEM 4.1.2

X	Exact Solution	Result of New Method	Error in New Method Order $P = 5$	Error in Adesanya [2] Order $P = 6$
0.1	0.0049999551874560100	0.00499995833058056099	$3.14312455 \times 10^{-09}$	$4.27300000 \times 10^{-08}$
0.2	0.01999865908023810000	0.01999866682345690130	$7.74321880 \times 10^{-09}$	$1.20759000 \times 10^{-06}$
0.3	0.04498987410259470000	0.04498987942740723500	$5.32481254 \times 10^{-09}$	$8.60719000 \times 10^{-06}$
0.4	0.07995737735167610000	0.07995737788388623047	$5.32210130 \times 10^{-10}$	$3.40900400 \times 10^{-05}$
0.5	0.12487004764653700000	0.12487005733490623182	$9.68836923 \times 10^{-09}$	$9.74068000 \times 10^{-05}$
0.6	0.17967712636121700000	0.17967714100994159450	$1.46487246 \times 10^{-08}$	$2.25711000 \times 10^{-04}$
0.7	0.24430361290038500000	0.24430361658620586118	$3.68582086 \times 10^{-09}$	$4.51454700 \times 10^{-04}$
0.8	0.31864597946467400000	0.31864600868048815589	$2.92158142 \times 10^{-08}$	$8.08472900 \times 10^{-04}$
0.9	0.40256860621313400000	0.40256861961077550241	$1.33976415 \times 10^{-08}$	$1.32622070 \times 10^{-03}$
1.0	0.49590033762933700000	0.49590038143998373466	$4.38106467 \times 10^{-08}$	$2.02205460 \times 10^{-03}$

TABLE 4.1.3: ABSOLUTE ERRORS COMPARING THE EXACT AND NUMERICAL SOLUTION OF HTEPM FOR PROBLEM 4.1.3

X	Exact Solution	Result of New Method	Error in New Method Order $P = 5$
0.1	0.170440346269364	0.17044034739217347603	$1.12280948 \times 10^{-09}$
0.2	0.141582173138664	0.14158217447659620875	$1.33793221 \times 10^{-09}$
0.3	0.113282963581607	0.11328296817008674018	$4.58847974 \times 10^{-09}$
0.4	0.085545249227360	0.0855454145299195369	$1.65302560 \times 10^{-08}$
0.5	0.0585436828645928	0.05854370999334741200	$2.71287546 \times 10^{-08}$
0.6	0.0325108774782471	0.03251091543736197638	$3.79591149 \times 10^{-08}$
0.7	0.00780685408274400	0.00780690667931403256	$5.25965700 \times 10^{-08}$
0.8	-0.151523368042584	-0.1515226448987367326	$7.23143847 \times 10^{-08}$
0.9	-0.359116451185857	-0.3591154842620219391	$9.66923835 \times 10^{-08}$
1.0	-0.540041077972614	-0.5400398437640223413	$1.23420860 \times 10^{-07}$

5.0DISCUSSION OF RESULTS

Problems 4.1.1 is a non-linear problem derived by Tanner to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface. Problem 4.1.2 considered Blassius equation in thermodynamics. The non-linear Genesio equation of problem 4.1.3 is a non-linear chaotic system from [15]. The results were displayed in Tables 4.1.1, 4.1.2 and 4.1.3 respectively. The absolute errors obtained from tables 4.1.1 and 4.1.2 revealed that on comparison with the exact solution, the low errors resulted demonstrate their effectiveness and accuracy as the schemes performed favorably well. The exact solution, however, for problems 4.1.1, 4.1.2, and 4.1.3 were not available. Hence, they were generated directly using Maple software environment

6.0 CONCLUSION

The construction of a new class of continuous implicit two-step hybrid scheme capable of solving Initial Value problems of third order ODEs has been the central concern in this work. The Orthogonal Polynomials valid in the interval $[-1,1]$ with respect to weight function $w(x) = 1 + \frac{x}{2}$ have been chosen as basis functions to develop the schemes using interpolation and collocation techniques with the incorporation of equally spaced off-step points in order to approximate the solutions of IVPs. The scheme is capable of handling non-linear application problems. Tables 1, 2 and 3 displays the accuracy of the numerical results of the HTEPM with the exact solution and existing methods. The desirability and superiority of the method have been established by the numerical results.

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