ECN 302: **INTERMEDIATE Micro-Economic THEORY**

The Theory of Demand

The concept of equilibrium is very important in economic analyses. It is a state in which forces making for change in opposing direction are perfectly in balance so that there is no tendency to change. Equilibrium may either be stable or unstable. A stable equilibrium is one where if there is any slight deviation from it, the forces acting will automatically ensure a restoration of the equilibrium. On the other hand, an unstable equilibrium is one where if there is any slight deviation from it, forces acting are such that will push the system further away from equilibrium.

**Market Equilibrium (Mathematical Representations)**

In an isolated market, partial equilibrium condition consists of equating supply and demand:

$Q_d = Q_s$ or $E = Q_d - Q_s = 0$

where E is excess demand.

Given the model $Q_d = a - bp \quad (a, b > 0)$

$$Q_s = -c + dp \quad (c, d > 0)$$

Solving for p and Q yields

$$\bar{p} = \frac{a+c}{b+c}; \qquad \bar{Q} = \frac{ad - bc}{b+d}$$

**Numerical examples:**

Given $Q_d = 18 - 2p$

$$Q_s = -6 + 6p$$

$$\bar{p} = 1; \qquad \bar{Q} = 12$$

The models may however be non-linear such as follows:

$$Q_d = 4 - p^2$$

$$Q_s = 4p - 1$$

$$\bar{p} = 1; \qquad \bar{Q} = 3$$

Note that equating $4 - p^2$ and $4p - 1$ yields a quadratic equating with two distinct roots 1 and -5. However, since a negative price does not have any economic meaning, -5 is disregarded.

When several interdependent commodities are simultaneous considered, equilibrium will require the absence of excess demand for each and every commodity include in the model. The presence of excess demand for one commodity will affect a other commodities. Hence, for the n-commodity case,

$$E_i = Q_{di} - Q_{si} = 0; \quad (i = 1, 2 \cdots n)$$

for the two-commodity case

$$Q_{d1} - Q_{s1} = 0, \quad Q_{d2} - Q_{s2} = 0$$

$$Q_{d1} = a_0 + a_1 p_1 + a_2 p_2$$

$$Q_{s1} = b_0 + b_1 p_1 + b_2 p_2$$
$$Q_{d2} = \alpha_0 + \alpha_1 p_1 + \alpha_2 p_2$$
$$Q_{s2} = \beta_0 + \beta_1 p_1 + \beta_2 p_2$$

First, the model is simplified by equating the demand and supply equations for each commodity so that.

$$(a_0 - b_0) + (a_1 - b_1)p_1 + (a_2 - b_2)p_2 = 0$$
$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)p_1 + (\alpha_2 - \beta_2)p_2 = 0$$

Let $c_1 = a_1 - b_i \ and \ \delta_i = \alpha_i - \beta_i$

Thus,

$$\begin{pmatrix} c_1 & c_2 \\ \delta_1 & \delta_2 \end{pmatrix}\begin{pmatrix} p_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} -c_0 \\ -\delta_0 \end{pmatrix}$$

$$\overline{p_1} = \frac{c_2\delta_0 - c_0\delta_2}{c_1\delta_2 - c_2\delta_1}; \qquad \overline{p}_2 = \frac{c_0\delta_1 - \delta_1\delta_0}{c_1\delta_2 - \delta_2\delta_1};$$

**Exercise:**

Find the solution for the following numerical model

1. $Q_{d1} = 10 - 2p_1 + p^2$

   $Q_{s1} = -2 + 3p_1$

2. $Q_{d2} = 15 + p_1 - p^2$

   $Q_{s2} = -1 + 2p_2$

**The Cobweb-Model**

The more realistic model is one which relates Qs in one time period to price in the previous period such that for the partial equilibrium system:

$$Q_d = a - bp_t \quad (a, b > 0)$$
$$Q_s = -c + dp_{t-1} \quad (c, d > 0)$$

The solution requires the method of solving difference equation. The standard first-order difference equation is of the form

$$Y_{t+1} + ay_t = c$$

with the definite solution

$$Y_t = \left(Y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a} \qquad (for\ a \neq -1)$$

$$Y_t = Y_0 + c \qquad (for\ a = -1)$$

Equating $Q_d$ and $Q_s$

$$a - bp_t = -c + dp_{t-1}$$

$$bp_t + dp_{t-1} = a + c$$

or $$bp_{t+1} + dp_t = a + c$$

Normalizing

$$p_{t+1} + \left(\frac{d}{b}\right)p_t = \frac{a+c}{b}$$

By direct substitution

$$p_t = p_0 \left(\frac{a+c}{b+d}\right)\left(-\frac{d}{b}\right)^t + \frac{a+c}{b+d}$$

Since $d, b > 0, then \frac{-d}{b} < 0,$ and we must have an oscillatory time path. Whether or not the system will be damped or explosive depends on the relative magnitudes of b and d.

**Exercise:**

Determine the time path of price in each of the following models and determine whether the systems are stable or not.

(i) $$Q_{dt} = 18 - 3p_t$$

$$Q_{st} = -3 + 4p_{t-1}$$

(ii) $$Q_{dt} = 20 - 2p_t$$

$$Q_{st} = -5 + 3p_{t-1} \qquad P_0 = 4$$

**Producers Surplus**

In general, given an inverse demand function of the form $p = a - bq$, the consumer's surplus can also be calculated when a certain quantity, q, is bought by finding the definite integral with zero and $q$ as the lower and upper limits of integration respectively and then subtracting the product of q and its corresponding price, *p* i.e.

$$\int_0^q (a - bq)dq - pq$$

The definite integral is given by the area of 0A Bq* while the product of P* and q* is given by the area of rectangle $0p^*Bq^*$. Consumer surplus is then given by the difference between the two areas.

For example, let the demand curve be given as

$q = 10 - p$

Consumer's surplus when P = 5 is calculated as follows

When $p = 5$,

$q = 5$

$$\int_0^5 (10 - q)\, dq = 10(5) - 0.5\,(5)^2 = 37.5$$

Total expenditure $= 5 \times 5 = 25$

Therefore, consumer's surplus is given as

37.5 – 25 = 12.5

The concept of consumer's surplus is of practical importance. For example, it can easily be used to analyze the loss to consumers due to restrictions on imports.

Restriction of imports will lead to a higher price and therefore a reduction in the quantity purchased. Before the restriction, the consumers' surplus is given by the triangle $P_1AC$. After the restriction, the consumers' surplus now reduces to triangle P1AB. The loss to the consumers is given by the area $P_1P_2BC$ which can be broken into two: rectangle $P_1P_2BD$ and triangle DBC. The rectangle is the increase in the cost the consumers have to pay for sugar, while the triangle represents the net loss to the consumers.

***Mathematical Derivation of the Slope of an Indifference Curve***

An indifference curve (for the two-commodity case) is given by the equation.

$$U = U(x, y) = a \qquad \ldots E4.7$$

where is a constant.

Taking the total differential, we obtain

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0$$

The slope of the curve then is

$$\frac{dy}{dx} = \frac{\partial u}{\partial x} \div \frac{\partial u}{\partial y} \quad or \quad \frac{-MU_X}{MU_Y}$$

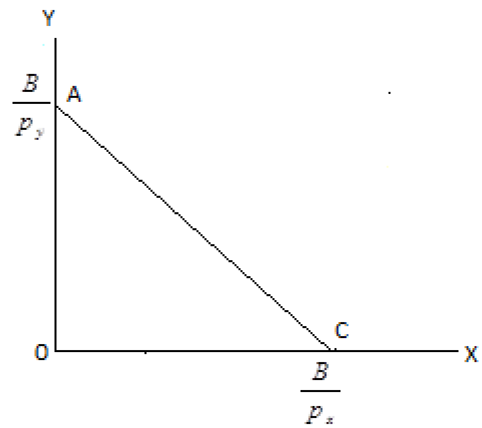
**The Budget Line**

The next step is to introduce the prices of commodities on the consumer's budget. A consumer's purchase can be determined once commodities' prices, the consumer's budget and taste are known. An advantage of the indifference curve is that all the three variables can be represented at once.

The budget is defined in mathematical form as:

$$B \geq P_x X + P_Y Y \qquad \ldots E4.2$$

Graphically, it appears as in figure 4.7. The straight line is called the budget line. The consumer can buy any combination of the two products along the line provided he is spending the whole budget (the case of equality sign in E4.3). He may also buy any combination within the triangles OAC if he is not spending the whole budget (the case of inequality sign in E4.2). The budget line can therefore be viewed as the boundary to the consumer's opportunities for acquiring *X* and *Y*.




**The budget line**

The slope of the budget line is the ratio of the prices of the two commodities, $\frac{P_x}{P_y}$. For any budget line,

Slope = $\frac{quantity\ of\ Y}{quantity\ of\ X} = \frac{budget}{P_y} \Big/ \frac{budget}{P_x} = P_x / P_y$

This may also be derived from the mathematic definition by making y subject of the formula such that the equation of the budget lines is

$$Y = \frac{B}{p_y} - \frac{p_x}{p_y} X \qquad \ldots E\,4.3$$

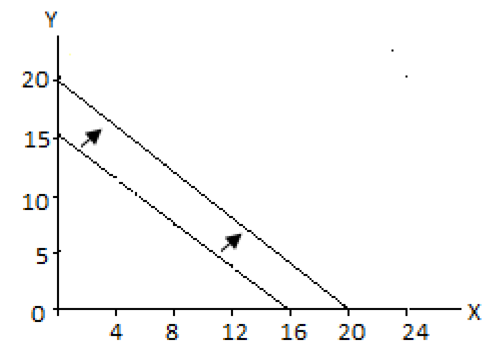
Let the price of *Y* be ₦4 a unit the price of *Y* ₦5. Let the consumer's budget be ₦60 per time period. Then 15 units of *Y* could be bought if the consumer spent the whole budget on *Y* and 12 units of *X* could be bought if the consumer spent his whole budget on it. The equation of the budget line is given as

$$Y = 15 - 1.25X \qquad \ldots E4.5$$

The position of the budget line depends on the size of the budget. If the budget increases the line would be further to the right. Changes in prices and the size of budget are shown by changing the slope and position of the budget line. If, in our example, the budget size increases to ₦80 while prices remain the same, the consumer will now be able to purchase 20 units of *Y* if the whole budget is spent on *Y* and 16 units of *X* if all is spent on it. The new budget line will be parallel, and to the right of the old one.

Its equation is

$$Y = 20 - 1.25X \qquad \ldots E4.6$$




**Effect of an Increase in Income**

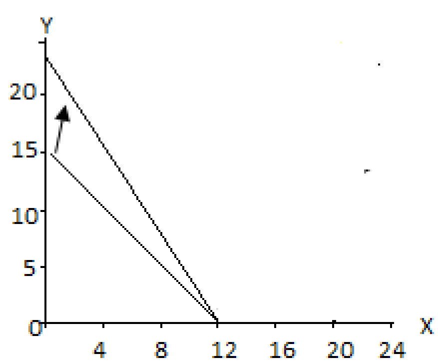
A reduction in the budget size with fixed prices will shift the curve inwards to the left and parallel to the old budget line.

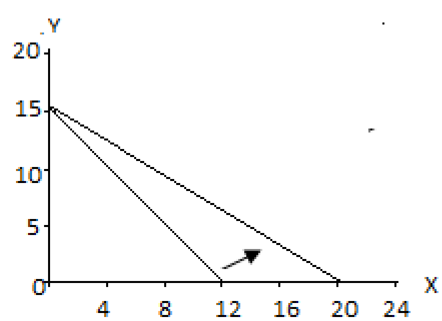
Suppose there is a fall in the price of one of the commodities, while the price of the other and the budget size remains the same, the effect is that more of the commodity whose price has fallen can be purchased if all income is spent on it. In our example suppose the price of *X* falls to ₦3 while that of *Y* remains at ₦5 and the budget remains at ₦60.

Equation of the new budget line is

$$Y = 5 - 0.2X \qquad \ldots E4.6$$

The student may trace the effects of an increase in price.



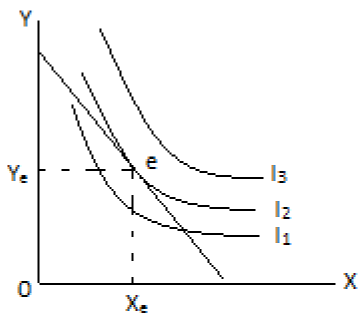
**Effect of a fall in Price of Commodity X**

**Effect of a fall in Price of Commodity Y**

**Equilibrium of the Consumer**

At the point of tangency, the slope of the indifference curve is the same as that of the budget line at that point.

i.e., $\frac{MU_y}{MU_y} = \frac{P_x}{P_y}$



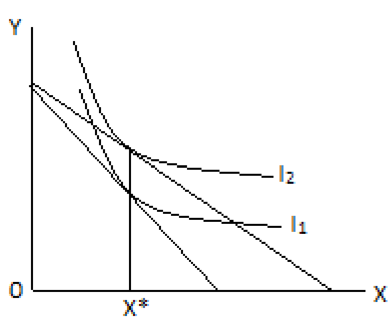

**Equilibrium of the consumer**

By some slight manipulations, this may be re-written as

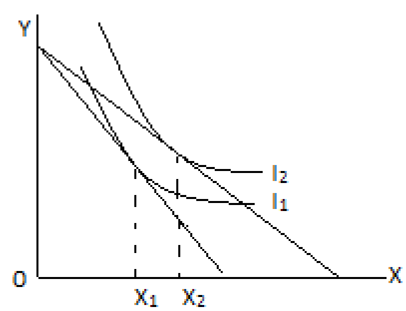
$$\frac{MU_x}{P_x} = \frac{MU_y}{P_y}$$

These imply that at equilibrium the MRS is equal to the ratio of prices of the two commodities, and equivalently, that the marginal utilities derived from the last naira spent on each commodity are equal. This last statement is equivalent to the propositions of the cardinal analysis.

**An Unchanged Quantity Curve**



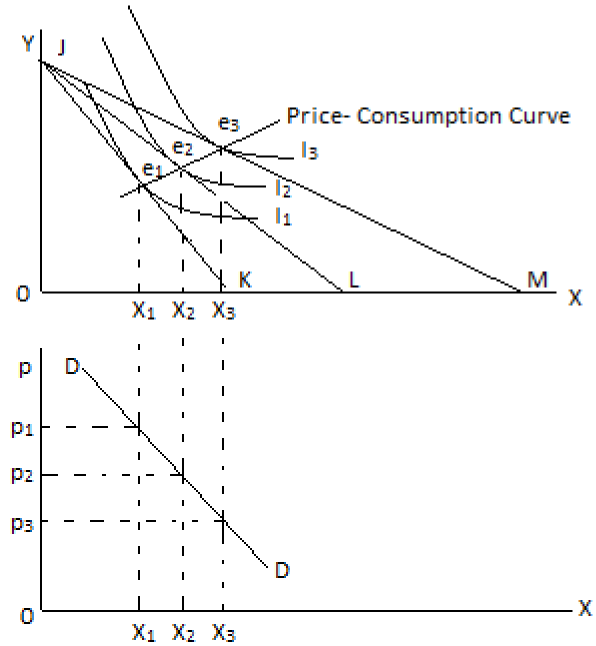

**A Reduced Quantity with a with a Higher Indifference Higher Indifference curve**




A mathematical determination of equilibrium requires the knowledge of the concept of Langragian multiplier. (Students may refer to a textbook on mathematics for Economists for this topic). The concept has an advantage of making it possible to analyse the case of more than two choice variables.

**Derivation of Demand Curve and the Price**

**Consumption Line**



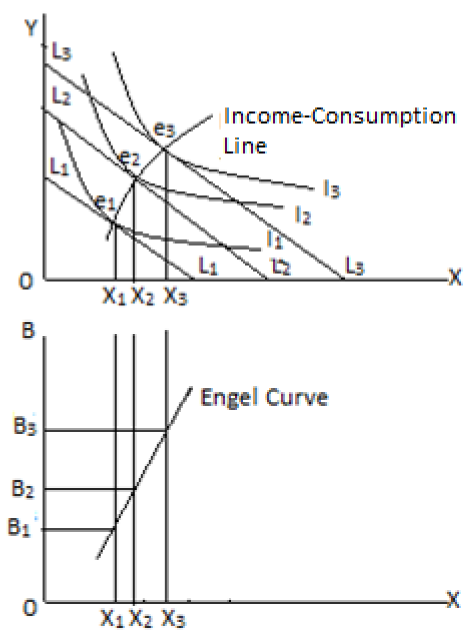

**Derivation of the Price-Consumption Line and**

**Demand Curve Using the Indifference Curve Analysis**

**Income-consumption Line (or Curve)**

**Engel Curve**

The income-consumption curve can be used to derive Engel curves, which are important for studies of family expenditure patterns. An Engel curve is the relationship between the equilibrium quantity purchased of a good and the level of income.



**Income and Leisure (An Application)**

Assume a consumer's utility depends on income Y and leisure L, then

$$U = g(L, Y).$$

The rate of substitution of income for leisure is

$$-\frac{dY}{dL} = \frac{g_L}{g_Y} \qquad \text{…E4.8}$$

if we denote the amount of work performed by W, the wage rate as r, and the total amount of time available as t, then be definition:

L = T – W and the budget constraint is:

Y = rW

By substitution,

$$U = g(T-W, rW)$$

$$\frac{dU}{dW} = g_L \frac{dL}{dW} + g_Y \frac{dY}{dW} = 0 \qquad \text{… E.4.9}$$

$$-g_L + g_Y r = 0$$

$$g_L / g_Y = r$$

$$\therefore \quad \frac{-dY}{dL} = g_L / g_Y = r \qquad \text{…E. 4.10}$$

This means that the rate of substitution of income for leisure equals the wage rate. In review of the fact that this last equation is a relation in terms of W and r, it is recognized as the supply curve for work which states how much the consumer will work at various wage rates. Alternatively, the equation provides the consumer's demand curve for income.

Given the utility function for a time period of one day as:

$$U = 48L + LY - L^2$$

Then $U = 48(T-W) + (T-W)Wr + (T-W)^2$

$$\frac{dU}{dW} = -48 - Wr + r\,(T-W) + 2(T-W) = 0$$

Therefore: $$W = \frac{T(r+2)-48}{2(r+1)} \qquad \text{E. 4.11}$$

And $$Y = \frac{[T(r+2)-48]r}{2(r+1)} \qquad \text{E. 4.12}$$

The SOC for maximization is satisfied since

$$\frac{d^2U}{dW^2} = -2\,(r+1) < 0$$

Clearly since T = 24 hours, if r = 0, W = 0. Since $dW/dr > 0$, hours worked will increase with the wage rate. Since $\ell im\, W = 12$, it follows that the worker will never work for more than 12 hours per day irrespective of how high the wage rate is.

It should be noted that these conclusions are based strictly on the specified model. Another sets of result may emerge from another type of model.

**Slutsky equation**

Given the utility function

$U = f(q_1\ q_2)$ and the budget constraint

$B = p_1 q_1 + p_2 q_2$

then the Largrangean function for optimization is specified as

$$L = f(q_1 q_2) + \lambda (B - p_1 q_1 - p_2 q_2) \qquad \text{…E4.13}$$

$$L_1 = f_1 - \lambda p_1 = 0$$

$$L_2 = f_2 - \lambda p_2 = 0 \qquad \text{.. E4.14}$$

$$L_\lambda = B - p_1 q_1 - p_2 q_2$$

Total differentiation of each of these yields

$$f_{11} dq_1 + f_{12} dq_2 - p_1 d\lambda - \lambda dp_1 = 0$$

$$f_{21} dq_1 + f_{22} dq_2 - p_2 d\lambda - \lambda dp_2 = 0 \qquad \text{…E4.15}$$

$$dB - dp_1 q_1 - q_1 dp_1 - p_2 dq_2 - q_2 dp_2 = 0$$

Transferring those terms involving changes in the exogenous variable to the right hand side,

$$f_{11} dq_1 + f_{12} dq_2 - p_1 d\lambda - \lambda dp_1$$

$$f_{21} dq_1 + f_{22} dq_2 - p_2 d\lambda - \lambda dp_2 \qquad \text{…E416}$$

$$-p_1 dq_1 - p_2 dq_2 \quad = \quad q_1 dp_1 + q_2 dp_2 - dB$$

This is represented in the matrix form as

$$\begin{pmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_2 & 0 & d\lambda \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \\ d_1 \end{pmatrix} = \begin{pmatrix} \lambda dp_1 \\ \lambda dp_2 \\ dp_1 + q_2 dp_2 - dB \end{pmatrix}$$

Let $dp_1 = dp_2 = 0,$ but keep $dB \neq 0$ and divide through by

$$\begin{pmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} dq_1 / dB \\ dq_2 / dB \\ d\lambda / dB \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Solution by Cramer's rule yields

$$\frac{dq_1}{dB} = \left|\frac{1}{J}\right| \begin{vmatrix} f_{12} & -p_1 \\ f_{22} & -p_2 \end{vmatrix} \text{ and } \qquad \frac{dq_2}{dB} = \left|\frac{1}{J}\right| \begin{vmatrix} f_{11} & -p_1 \\ f_{12} & -p_2 \end{vmatrix}$$

$|J|$ is the determinant of the Jacobian, the lead matrix on the left hand side.

By the SOC, $|J| = |H| > 0.\ Also\ p_1\ and\ p_2$ are positive.

However, without additional information about the relation magnitude of $p_1$ and $p_2$ and the $f_1$'s we cannot ascertain the signs of these comparative static derivatives. This means that as the consumer's budget increase, additional purchases of the commodities may either increase or decrease. If the optimal purchase of any of the commodities decrease with income, the commodity is said to be an inferior good as against a normal one.

Next, let $dp_1 \neq 0 \; while \; dp_2 = dB = 0$ and divide Through by $dp_1$.

$$\begin{pmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} dq_1/dp_1 \\ dq_2/dp_1 \\ d\lambda/dp_1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ q_1 \end{pmatrix}$$

Thus,

$$\frac{dq_1}{dp_1} = \frac{\lambda}{J} \underbrace{\begin{vmatrix} f_{22} & -p_2 \\ -p_2 & 0 \end{vmatrix}}_{T_1} + \frac{q_1}{|J|} \underbrace{\begin{vmatrix} f_{12} & -p_1 \\ f_{22} & -p_2 \end{vmatrix}}_{T_2}$$

and

$$\frac{dq_2}{dp_2} = \frac{\lambda}{|J|} \underbrace{\begin{vmatrix} f_{21} & -p_2 \\ -p_2 & 0 \end{vmatrix}}_{T_3} + \frac{q_1}{|J|} \underbrace{\begin{vmatrix} f_{11} & -p_1 \\ f_{21} & -p_2 \end{vmatrix}}_{T_4}$$

How do we interpret these results?

$\frac{dq_1}{dp_1}$ is the measure of the change in $q_1$ as a result of a change in its price. It has two components. First, note that $T_2 = \frac{dq_1}{dB}$ with $q_1$ as a weighting factor. It is therefore the income effect of a price change. Obviously, the more prominent the place of $q_1$ in the total budget, the greater the income effect will be $T_1$ is the substitution effect. Assume that in the third equation, $dB = dp_2$, then it becomes

$$-p_1 \, dq_1 - p_2 \, dq_2 = q_1 \, dp_1$$

Since the effectual income change is in the term $q_1 dp_1$, to remove this effect, we set the expression equal to zero to zero so that the vector of constraints in 6 becomes $\begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$ and

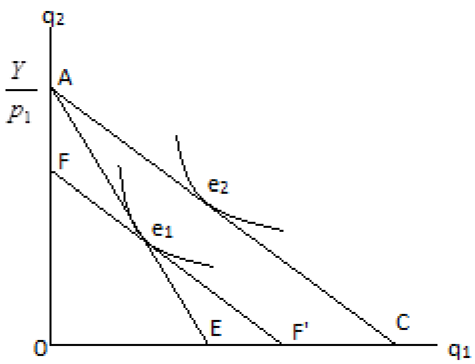
$$\left|\frac{dq_1}{dp_1}\right| less\,income\,effect = \frac{1}{|J|}\begin{vmatrix} \lambda & f_{12} & -p_1 \\ 0 & f_{22} & -p_2 \\ 0 & -p_2 & 0 \end{vmatrix} = \frac{\lambda}{|J|}\begin{vmatrix} f_{22} & -p \\ -p_2 & 0 \end{vmatrix} = T_2$$

Hence $\dfrac{dq_1}{dp_1} = \left|\dfrac{dq_1}{dp_1}\right| \qquad\qquad \dfrac{-dq_1}{dB}\; q_1$

Less than substitution effect $\qquad$ Income effect

This result is known as the Slutsky equation. The substitution effect is sometime called the change in compensated demand because the consumer is being compensated for a price rise by having enough income given back to him to purchase his old bundle.

Calculating the Substitution Effect




Let us calculate how much we have to adjust money income so as to keep the old bundle just affordable. Let B be the amount of money income that will just make the original consumption bundle affordable; this will be the amount of money income associated with the pivoted budget line. Since $(\bar{q}_1, \bar{q}_2)$ is affordable at both $(p_1, p_2)$ *and* $(p_1', p_2, B')$, we have

$$B' = p_1'\, q_1 + p_2\, q_2$$

$$B = p_1\, q_1 + p_2\, q_2$$

Subtracting the second equation from the first

$$B' - B \quad q_1(p_1' - p_1) \qquad\qquad \ldots \text{E4.18}$$

This equation says that the change in income necessary to make the old bundle affordable at the new prices is just the original amount of consumption of good 1 times the change in prices.

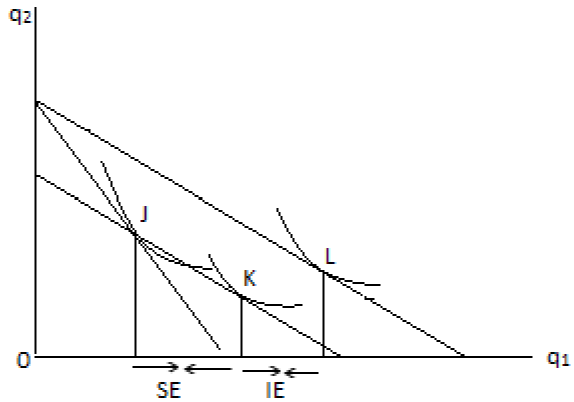
Letting $dp_1 = p_1' - p_1$ represent the change in price 1, and $dB = B' - B$ represent the change in income necessary to make the old bundle just affordable, we have

$$dB = q_1 dp_1 \qquad\qquad \ldots \text{E4.19}$$

Note that the change in income and the change in price will always move in the same direction: if the price goes up, then we have to raise income to keep the same bundle affordable.

Suppose that a consumer is originally consuming 20 units of a commodity per week and that each unit cost ₦5. If the price of the commodity rises by 10 per cent, so that $\Delta p_1$ = ₦6 – N5 = ₦1- how much would income have to change to make the old consumption bundle affordable? In terms of our formula,

$$dB \quad = \quad dp_1 \; x \; q_1 \quad = 1 \; x \; 20 = N20.00$$




The movement from J to K is the substitution effect which indicates how the consumer substitutes one good for the other when price changes but purchasing power remains constant.

More precisely, the substitution effect $dq_1^2$ , is the change in the demand for good 1 when the price of good 1 changes from $p_1$to $p'_1$ and at the same money income changes to Y'.

$$dq_1^s \quad = \quad q_1\,(p_1', Y') - q_1\,(p_1, Y) \qquad \ldots \text{ E 2.20}$$

To order to determine the substitution effect, we must use the consumer's demand function to calculate the optimal choices at $(p_1', Y')$ *and* $(p_1, Y)$.The change in the demand for good 1 may be large or small, depending on the shape of the consumer's indifference curves.

Example:

Suppose that consumer has a demand function of the form

$$q_1 \quad = \quad 10 + \frac{Y}{10p_1}$$

Originally his income is N120 per week and the price N3 per unit. Thus the demand will be

$$10 + \frac{Y}{10p_1} \quad = \quad 14 \, units \; per \; week$$

Now suppose the price falls to N2 per unit. Then his demand at this new price will be $10 + 120/(10 \times 2) = 16$ units order to calculate by how much income would have to change in order to make the original consumption just affordable when the price is N2, we apply the formula.

$$dY \quad = \; q_1 \; dp_1 = 14 \; x \; 2 - 3) = -N14$$

Thus, the level of income necessary to keep purchasing power constant, $Y' = Y + dY = 120 - 14 = N106$ .

What is the consumer's demand at the new price, N2 per unit, and this level of income? Just plug the numbers into the demand function to find.

$$q_1\ (p', Y') = q_1\ (2,106) = 10 + \frac{106}{10 \times 3} = 15.3$$

Thus, the substitution effect is

$$dq_1^s = q_1\ (2,106) - q_1\ (3,120) = 15.3 - 14 = 1.3$$

The income effect is

$$dq_1^Y = q_1\ (2,120) - (2,106) = 16 - 15.3 = 0.7$$

**Mathematical appendix**

(i) Derivation of the slope of an indifference curve

An indifference curve (for the two-commodity case) is given by the equation

$$U = (x, y)\ a \qquad \text{…E 4.21}$$

where a is a constant.

Taking the total differential, we obtain

$$dU = \frac{\partial U}{\partial x}\ dx + \frac{\partial U}{\partial y}\ dy\ = 0$$

The slope of the curve then is

$$\frac{dy}{dx} = \frac{\partial U}{\partial x} \div \frac{\partial U}{\partial y} \text{ or } \qquad \frac{-MU_X}{MY_Y}$$

(ii) Derivation of Equilibrium

Maximise

$$U = U(x, y) \qquad \text{(objective function)}$$

subject to

$$B = Xp_x + Yp_y \qquad \text{(the constraint)}$$

The Lagragean function is formulated as

$$L = U(x, y) + \lambda(B - Xp_x - Yp_y) \qquad \text{… E4.22}$$

The first order condition (necessary condition) for maximum is derived as

$$L_x \ = \ U_x - p_x = 0 \qquad \text{… E4.23}$$

$$L_y \ = \ U_y - p_y = 0 \qquad \text{… E4.24}$$

$$L_\lambda \ = \ B - Xp_x - Yp_y = 0 \qquad \text{… E4.25}$$

Re-arrangement equations E4.23 and E4.24, and dividing E4.23 by E4.24.

$$\frac{U_x}{U_y} \ = \ \frac{MU_x}{MY_y} \ = \ \frac{p_x}{p_y}$$

Suppose the utility function is stated more explicitly as

$$U \ = \ XY \ and \ B = Xp_x + Yp_y$$

$$L \ = \ XY + \lambda(B - Xp_x - Yp_y) \qquad \text{…E4.26}$$

$$L_x \ = Y - \lambda p_x = 0 \qquad \text{… E4.27}$$

$$L_y \quad = X - \lambda p_y = 0 \qquad \ldots \text{ E4.28}$$

$$L \quad = \ B - Xp_x - Yp_y = 0 \qquad \ldots \text{ E4.29}$$

Solving the first order condition equations simultaneously yields

$$X \quad = \frac{B}{2p_x}, \qquad Y \quad = \frac{B}{2p_y}$$

These are two demand curves that are contingent on continued optimizing behaviour by the consumer. The demand curves have two properties namely:

(i) they are single-valued functions of prices and income, i.e a single maximum and therefore, a single commodity combination corresponds to a given set of prices and income; and

(ii) they are homogeneous of degree zero, i.e if all prices and the consumers' money income and increased in the same proportion, the quantities demanded do not change. This means that if the consumer's money income increases, but prices change proportionately, he will not behave as if he is richer. In order words, the consumer not suffer from money illusion.

## THE THEORY OF PRODUCTION

### Output Elasticity

Out elasticity of labour denoted as

$$\omega_L = \frac{\partial(\ln q)}{\partial(\ln L)} \quad = \frac{\partial q}{\partial L} \times \frac{L}{q} = \frac{MP_L}{AP_L}$$

$\omega_L$ could be greater than 1, equal to unity or lie between 1 and zero depending on the relative magnitudes of the MP and AP.

### Elasticity of Substitution

If a production function has convex isoquants, the RTS of L for K and the input ratio K/L will decline as L is substituted for K along an isoquant. The elasticity of substitution σ is a pure number that measures the rate at which substitution takes place, and is defined as the proportionate rat of change of the input, and is defined as the proportionate rate of change of the input ratio divided by the proportionate rate of change of MRTS.

$$\sigma = \frac{d\ln(K/L)}{d\ln(f_L/f_K)} \quad = \quad \frac{d(K/L)}{d(f_L/f_K)} \times \frac{f_L/f_K}{K/L}$$

Given the Cobb-Douglas production function

$$q \quad = \ AK^aL^b$$

$$MP_L = \ bAK^aL^{b-1} \quad = \ b(AK^aL^b)L^{-1} = bAP_L$$

$$MP_K = \ aAK^{a-1}L^b \quad = \ a(AK^aL^b)K^{-1} = aAP_K$$

Given that

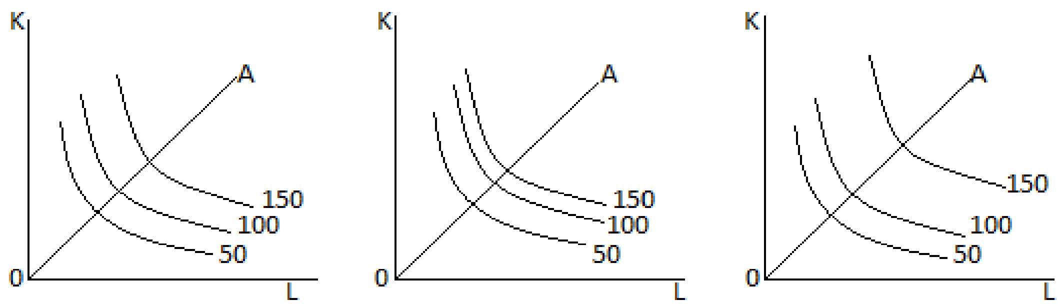
$$RTS_{L,K} = f_L / f_K = \frac{b\,q/L}{a\,q/K} = \frac{b}{a}\frac{K}{L}$$

Substitute E5.9-11 in E5.8, then

$$\sigma = \frac{d\left(\frac{K}{L}\right)}{d\left(\frac{bK}{aL}\right)} \times \frac{\frac{bK}{aL}}{\frac{K}{L}} = \frac{d\left(\frac{K}{L}\right)}{\frac{b}{a}d\left(\frac{K}{L}\right)} \times \frac{\frac{bK}{aL}}{\frac{K}{L}} = 1$$

This shows that the coefficient of elasticity of substitution for a Cobb-Douglas production function is one.

**Increasing, Decreasing and Constant Returns to Scale**




**Constant Returns** **Increasing Returns** **Decreasing Returns**

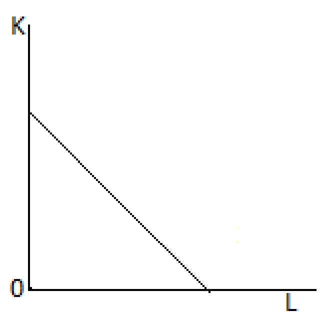
**Isocost**

The other tool that is employed in this analysis is the isocost, the consumer's budget line equivalent in the theory of production. The isocost function indicates those factor combinations which result in the same cost or require the same expenditure by the firm. The isocost line is defined by the equation.

where w (wage rate) and r (rate of interest) are the prices of labour and capital respectively. The equation can be re-arranged in the form of a straight line equation.

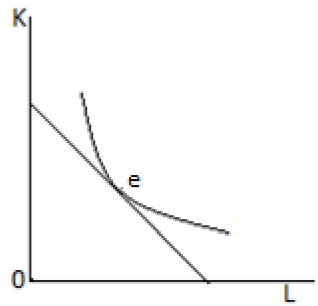
$$K = \frac{C}{r} - \frac{w}{r}L$$

In which case, $\frac{w}{r}$, the ratio of prices is the slope of the line




**Isocost**

To arrive at the equilibrium position of the firm, the map of isoquant is superimposed on the isocost line. Equilibrium is then determined at the point of tangency of the isocost with one of the isoquants. At the point tangency the slope of the isoquant and that of the isocost are equal, so that $\frac{MP_L}{MP_K} = \frac{w}{r}$. This may also be written as $\frac{MP_L}{w} = \frac{MP_K}{r}$.




**Equilibrium of the Firm**

This implies that a naira worth of labour yields the same addition to total product as a naira worth of labour. Suppose $\frac{MP_L}{w} > \frac{MP_K}{r}$ to attain equilibrium more units of labour and less units of capital will have to be employed. By so doing, $MP_L$ will fall while $MP_K$ will rise until the two sides of the equation are equal.

The equilibrium position represents two types of optimizing behaviour - output maximization subject to the cost constraint and cost minimization subject to output constraint. These two optimizing behaviour are, however, two different ways of describing the same issue as they will always yield the same result. As in the case of consumer behaviour, the analysis for more than two choice variables can only be easily examined with the technique of Langragian function.

When more than two variable inputs are involved, the optimum choice of quantities of many inputs can be indicated by extending the two-variable analysis. Let there be n inputs, then at equilibrium.

$$\frac{MP_1}{P_1} = \frac{MP_2}{P_2} = \ldots = \frac{MP_n}{P_n}$$

**OPTIMIZING BEHAVIOUR**

The optimizing behaviour of the firm may be analyzed as follows:

**Constrained Output Maximization**

The objectives is to maximize $q = f(L, K)$ subject to

$$C^0 = wL + rK + b$$

The Lagrangean function is given as

$$\phi = f(L,K) + \mu(C^0 - wL - rK - b) \quad \ldots\text{E5.8}$$

$$\phi_L = f_L - \mu w = 0 \quad \ldots\text{E5.9}$$

$$\phi_K = f_K - \mu r = 0 \quad \ldots\text{E5.10}$$

$$\phi_\mu = C^0 - wL - rK = 0 \qquad \text{…E5.11}$$

$$\frac{f_L}{f_K} = \frac{w}{r} \qquad \text{… E5.12}$$

i.e the ratio of the MP's of L and K must be equated with the ratio of the prices. It is also true from these FOC's that.

$$\mu = \frac{f_L}{w} = \frac{f_K}{r}$$

µis the derivative of output with respect to cost with prices constant and quantities variable. Assuming cost is variable; i.e. no fixed cost,

$$dC = wdL + rdK$$

from E5.9 and E5.10, $w = \frac{f_L}{\mu}$ and $r = \frac{f_K}{\mu}$

Therefore $dC = \frac{1}{\mu}(f_L dL + f_K dK)$

But $dq = f_L + f_K dK$. Therefore, $\frac{dq}{dC} = \mu$

The SOC requires that the relevant bordered Hessian determinant be positive.

$$\begin{vmatrix} f_{11} & f_{LK} & -w \\ f_{KL} & f_{KK} & -r \\ -w & -r & 0 \end{vmatrix} > 0$$

Constrained Cost Minimization

The objectives here is to minimize $C = wL + rK$ subject to $q^0 = f(L, K)$

$$Z = wL + rK + \lambda(q^0 - f(L, K) \qquad \text{…E5.13}$$

$$Z_L = w - \lambda f_L = 0 \qquad \text{…E5.14}$$

$$Z_K = r - \lambda f_K = 0 \qquad \text{…E5.15}$$

$$Z_\lambda = q^0 - f(L, K) = 0 \qquad \text{… E5.16}$$

These FOC's imply that $\frac{f_L}{f_K} = \frac{w}{r}$ and $\frac{f_L}{f_K} = \frac{w}{r}$

$\lambda$ is the reciprocal of the multiplier µ or the derivative of cost with respect to output.

The SOC requires that

$$\begin{vmatrix} -\lambda_{LL} & -\lambda f_{LK} & -f_L \\ -\lambda f_{KL} & -\lambda f_{KK} & -f_K \\ -f_L & -f_K & 0 \end{vmatrix} < 0$$

Given that $\frac{f_L}{\lambda} = w$ and $\frac{f_K}{\lambda} = r$, the last column and the last row are changed such that

$$\begin{vmatrix} -\lambda f_{LL} & -\lambda f_{lK} & \frac{w}{\lambda} \\ -\lambda f_{KL} & -\lambda f_{KK} & \frac{r}{\lambda} \\ \frac{-w}{\lambda} & \frac{-r}{\lambda} & 0 \end{vmatrix} < 0$$

Multiply the first and second column each by $\frac{-1}{\lambda}$ and then the last row by $-\lambda^2$, then

$$-\lambda \quad = \quad \begin{vmatrix} f_{11} & f_{LK} & -w \\ f_{KL} & f_{KK} & -r \\ -w & -r & 0 \end{vmatrix} \quad < \ 0$$

Since,

$$\lambda > 0, \quad = \quad \begin{vmatrix} f_{11} & f_{LK} & -w \\ f_{KL} & f_{KK} & -r \\ -w & -r & 0 \end{vmatrix} \quad > \ 0$$

Thus, the SOC is the same as that of constrained output maximization

**Profit Maximization**

An entrepreneur will rather maximize profit than minimize cost or maximize output.

The profit of a firm is the difference between the total revenue and his total cost.

$$\pi \quad = \quad pq - C$$

$$= \quad pf(L, K) - wL - rK - b \qquad \ldots E5.17$$

Profit is a function of the inputs so that to maximize profit we different the profit function with respect to the input.

$$\frac{\partial \pi}{\partial L} = pf_L - w = 0 \qquad \text{i.e. } pf_L = w$$

$$\frac{\partial \pi}{\partial K} = pf_K - r = 0 \text{ i.e. } pf_K = r$$

That is, the value of the marginal product of each factor must be equal to its price.

The SOC requires that the principal minors of the relevant Hessian determinant alternate in sign.

$$\frac{\partial^2 \pi}{\partial L^2} = pf_{LL} < 0; \quad \frac{\partial^2 \pi}{\partial K^2} = pf_{KK} < \ 0$$

and

$$\begin{vmatrix} \dfrac{\partial^2 \pi}{\partial L^2} & \dfrac{\partial^2 \pi}{\partial L \partial K} \\ \dfrac{\partial^2 \pi}{\partial K \partial L} & \dfrac{\partial^2 \pi}{\partial K^2} \end{vmatrix} = P^2 \begin{vmatrix} f_{LL} & f_{LK} \\ f_{KL} & f_{KK} \end{vmatrix} > 0$$

The first conditions imply that profit must be decreasing with respect to further applications of either L or K, while the second condition ensures that profit is decreasing with respect to further application of both L and K.

## Monopoly Equilibrium

Given the revenue function as

$$R = R(q) \qquad \ldots E7.1$$

and the cost function as

$$C = C(q) \qquad \ldots E7.2$$

the profit function π, is given as

$$\pi = R(q) - C(q) \qquad \ldots E7.3$$

Profit maximization requires that the FOC equals zero

i.e. $\dfrac{d\pi}{dq} = R'(q) - C'(q) = 0 \qquad \ldots E7.4$

i.e. $R'(q) = C'(q)$

$MR = MC$

Under perfect competition, since marginal revenue and average revenue are identical, the condition for profit maximization may be stated as

$AR = MC$

The SOC is satisfied if

$$R''(q) < C''(q) \qquad \ldots E7.5$$

i.e. the rate of change of marginal revenue with respect to output is less than that of marginal cost. Under perfect competition,

$R''(q) = 0, \quad while\, C''(q) > 0$.

Under monopolist $R''(q) < 0 \quad while\, C''(q) > 0$

Suppose the average revenue and total cost functions for a monopolist are given as

$P = 110 - 2q$

and $C = 25 + 10q$ respectively

For $MR = MC,$

$110 - 4q = 10$

$\therefore\ q = 25,\ P = 60,\ and\ \pi = 1225.$

For $AR = MC,$

$110 - 2q = 10$

$\therefore\ q = 50,\ P = 10,\ \pi = 25.$

It follows therefore that output is lower, price is higher and profits larger in monopoly than in perfect competition

## PRICE DISCRIMINATION

**Some Mathematical Derivation**

The profit function of the discriminating monopolist is given as

$$\pi = R_1(q_1) + R_2(q_2) - C(q_1 + q_2)$$

To maximize profit,

$$\frac{\partial \pi}{\partial q_1} = R'(q_1) - C'(q_1 + q_2) = 0$$

$$\frac{\partial \pi}{\partial q_2} = R'(q_2) - C'(q_1 + q_2) = 0$$

i.e. $R'(q_1) = R'(q_2) = C'(q_1 + q_2)$

The marginal revenue in each market must be equal to marginal cost of the output as a whole. If they are not equal, he can shift some products from the market with a higher revenue to the market with a lower marginal revenue.

Second order condition requires that the principal minor of the relevant Hessian determinant alternate in signs starting with a negative sign i.e. Given

$$\begin{vmatrix} R_1'' - C'' & -C'': \\ -C'' & R_2'' - C'' \end{vmatrix}$$

$$(R_1'' - C'') < 0 \ and \ (R_1'' - C'')(R_2'' - C'') - (-C'')^2 > 0$$

Suppose the monopolist is able to separate his consumers into two distinct markets such that $P_1 = 80 - 5q_1$

$$R_1 = 80q_1 - 5q_1^2$$

$$P_2 = 180 - 20q_2;\ R_2 = 80q_2 - 20q_2^2$$

$$C = 50 + 20(q_1 + q_2)$$

Note that total demand at any price P is the sum of the demands in the two markets.

Setting the MR in each market equal to MC of output as a whole

$$80 - 10q_1 = 20; \qquad 180 - 40q_2 = 20;$$

Solving for $q_1$ and $q_2$ and hence $P_1$ and $P_2$,

$$q_1 = 6, \qquad P_1 \ \ 50$$

$$q_2 = 4, \qquad P_2 \ \ 100$$

for the second order condition

$$R_1'' - C'' = -10 < 0, \begin{vmatrix} R_1'' & -C & -C \\ -C'' & R2 & -C \end{vmatrix} = \begin{vmatrix} -10 & 0 \\ 0 & -40 \end{vmatrix} = 400 > 0$$

$\pi = 450$ compare with 350 under no discrimination. The higher the number of possible divisions, the higher the profit that can be made.

**Perfect Discrimination**

The consumer pays less than he is actually willing to. The difference between what he will be to ready to pay and what he actually pays is the consumer's surplus (Section 2.7). The perfectly discriminating monopolist separates his market into such fine distinct unit that he is able to extract the total amount the consumer is willing to pay.

The revenue function then is

$$R = \int_0^{q^*} f(q)dq \qquad \ldots E7.6$$

and the profit function is

$$\pi = \int_0^{q^*} f(q)dq - C(q)$$

$$\frac{d\pi}{dq} = f(q) - C(q) = 0 \qquad \ldots E7.7$$

i.e. AR = MC

(Note here that unlike under perfect competition where
MR = AR = MC, the condition here is that
MR < AR = MC)

Using the numerical example on page 108,

$$\pi = \int_0^{q^*} (110 - 2q)dq - (25 + 10q)$$

$$\frac{d\pi}{dq} = 100 - 2q - 10 = 0$$

$$\therefore \quad q = 50, \pi = \quad 2475$$

**DUOPOLY EQUILIBRIUM**

**Mathematical Derivation**

Given the basic assumption that each duopolists maximizes his profit on the assumption that the quantity produced by his rival is invariant with respect to his own quantity decision, then the First Order Conditions are

$$\frac{\partial \pi_1}{\partial q_1} = \frac{\partial R_1}{\partial q_1} - \frac{\partial C_1}{\partial q_1} = 0 \qquad \ldots E7.11$$

$$\frac{\partial \pi_2}{\partial q_2} = \frac{\partial R_2}{\partial q_2} - \frac{\partial C_2}{\partial q_2} = 0 \qquad \ldots E7.12$$

The Second Order Conditions are

$$\frac{\partial^2 R_1}{\partial q_1{}^2} - \frac{\partial^2 C_1}{\partial q_1{}^2} < 0 \qquad \ldots E7.13$$

$$\frac{\partial^2 R_2}{\partial q_2{}^2} - \frac{\partial^2 C_2}{\partial q_2{}^2} < 0 \qquad \ldots E7.1$$

i.e. the MR must be increasing less rapidly than the MC for each producer.

The equilibrium output for each of the duopolists is derived by solving the FOCs for $q_1$ and $q_2$. Before solving for $q_1$ and $q_2$, however, the reaction functions which express the output of each duopolist as a function of his rival's output are determined from the FOCs such that from E.7.11 respectively we derive

$$q_1 = \psi(q_2) \qquad \ldots E7.15$$

and $$q_2 = \psi(q_1) \qquad \ldots E7.16$$

Assuming a linear market demand function so that

$$P = a - b(q_1 + q_2)$$

and cost functions are

$$C_1 = a_1 q_1 + b_1 q_1^2 \qquad \ldots E7.17$$

$$C_2 = a_2 q_2 + b_2 q_2^2 . \qquad \ldots E7.18$$

Then the profit functions are

$$\pi_1 = aq_1 - b(q_1 + q_2)\, q_1 - a_1 q_1 - b_1 q_1^2 \qquad \ldots E7.19$$

$$\pi_2 = aq_2 - b(q_1 + q_2)\, q_2 - a_2 q_2 - b_2 q_2^2 \qquad \ldots E7.20$$

The FOCs are

$$\frac{\partial \pi_1}{\partial q_1} = a - b(2q_1 + q_1) - a_1 - 2b_1 q_1 = 0 \qquad E7.21$$

$$\frac{\partial \pi_2}{\partial q_2} = a - b(q_1 + 2q_2) - a_2 - 2b_2 q_2 = 0 \qquad \ldots E7.22$$

Thus, the corresponding reaction functions are

$$q_1 = \frac{a - a_1}{2(b + b_1)} - \frac{b}{2(b + b_1)}\, q_2 \quad . \qquad \ldots E7.23$$

$$q_2 = \frac{a - a_2}{2(b + b_2)} - \frac{b}{2(b + b_2)}\, q_1 \quad . \qquad \ldots E7.24$$

Since b, $b_1$ and $b_2$ are all positive, a rise of either duopolist output will cause a reduction of the other's optimum output. Equilibrium output are derived by solving these last equations simultaneously.

Suppose $P = 100 - 0.5(q_1 + q_2), C_1 = 5q_1,$

$$C_2 = 0.5 q_2^2$$

The reaction functions are

$$q_1 = 95 - 0.5q_2 ; \quad q_2 = 50 - 0.25q_1$$

and the equilibrium solution is

$$q_1 \;=\; 80, q_2 \;=\; 30,\; p = 45, \pi_1 = 200, \pi_2 \;= 90$$

**Mathematical examples**

Assume firm A is the leader; it will substitute B's reaction function in its own profit function;

$$\pi_1 \;=\; pq_1 - C_1 \;=\; 95q \;-\; 0.5q_1^2 \;-\; 0.5q_1 q_1.$$

Substitute $q_2 = 50 - 0.25q_1$ (B's reaction function)

and maximize $\pi_1 \;\; 70q_1 \;-\; 0.375q_1^2$

$$\frac{\partial \pi_1}{\partial q_1} = 70 \;-\; 0.75q_1 \;=\; 0$$

Thus, $q_1 \;=\; 93\tfrac{1}{3} \;\; and \;\; \pi_1 \;=\; 3266\tfrac{1}{3}$

With B as the follower

$$q_2 \;=\; 50 \;-\; 0.25q_1 \;= 26\tfrac{2}{-3}$$

and $\pi_1 \;=\; 100q_2 - 5q_1 q_2 \;-\; q_2^2 \;=\; 711\tfrac{1}{9}$

Assume now that Firm B is the leader, it will substitute A's reaction function in its own profit function;

$$\pi_2 \;=\; pq_2 \;-\; C_2 \;=\; 100q_2 \;-\; 0.5q_1 q_2 \;-q_2^2$$

Substitute $q_1 \;= 95 - 0.5q_2$ (A's reaction function) and maximize

$$\pi_2 \;=\; 52.5q_2 \;-\; 0.75q_2^2$$

$$\frac{\partial \pi_2}{\partial q_2} \;=\; 52.5 \;-\; 1.5q_2 \;=\; 0$$

Thus, $q_2 \;=\; 35, \;\; and \;\; \pi_2 \;=\; 918.75$

With A as the follower,

$$q_1 \;=\; 95 \;-\; 0.5q_2 \;= 77.5$$

and $\pi_1 \;=\; 95q_1 - 0.5q_1^2 \;-\; 0.5q_1 q_2 \;=\; 3003.125$